

Asymptotic density of zeros of half range generalized Hermite polynomials

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Abstract

We investigate the global density of zeros of generalized Hermite orthogonal polynomials, subject to certain truncated conditions on its weight. We shall give explicitly the global density of zeros under some asymptotic conditions on the weight. Moreover we compute the asymptotic of the total energy of the equilibrium position of the system of n movable unit charges in an external field determined by the weight of the generalized Hermite polynomials. We will see that for finite n the energy is in direct relationship with the zeros of the orthogonal polynomials.

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1 Introduction

Stieltjes [12], [13] considered the following electrostatic model. Fix two charges $(\alpha + 1)/2$ and $(\beta + 1)/2$ at $x = 1$ and $x = -1$, respectively, then put n movable unit charges at distinct points in $] - 1; 1[$. The question is to determine the equilibrium position of the movable charges when the interaction forces arise from a logarithmic potential. Stieltjes proved that the equilibrium position is attained at the zeros of the Jacobi polynomial $P_n^{(\alpha; \beta)}(x)$. For a proof see Szego's book [11]. Another electrostatic problem is to have a fixed point charge $(\alpha + 1)/2$ at $x = 0$ and n movable unit point

charges at distinct points in $[0; 1[$. The state of equilibrium in the presence of an additional external potential $v(x) = x$ is now reached at the zeros of the Laguerre polynomial $L_n^{(\alpha)}(x)$ provided that the point charges interact according to a logarithmic potential.

Let $a \geq 0$, $a < b$, and consider the electrostatic model, where we put an n movable unit point charge at distinct points in $\Sigma_a = [a, +\infty[$ or $]-\infty, -a] \cup [a, +\infty[$. The state of the equilibrium in the presence of an additional external potential $V_n(x) = x^2 + 2\mu_n \log \frac{1}{|x|} + \log A_n(x)$ is realized at the zeros of the generalized Hermite polynomials $H_n^{\mu_n}(x, a)$ provided that the point charges interact according to a logarithmic potential, where $H_n^{\mu_n}(x, a)$ are the orthogonal polynomials with respect the weight function $w_{\mu_n}(x) = C|x|^{2\lambda_n}e^{-x^2}$ restricted to Σ_a , $A_n(x)$ is some functions which will be given later and C is a normalizing constant, see for instance [10]. In other words, for $(x_1, \dots, x_n) \in \Sigma_a$, let

$$E(x_1, \dots, x_n) = \sum_{i=1}^n V_n(x_i) + 2 \sum_{i < j} \log \frac{1}{|x_i - x_j|},$$

be the energy created by the n movable unit charge (x_1, \dots, x_n) in Σ_a . Then as it has been proved in [10], the equilibrium position is realized at $x_1^{(n,a)}, \dots, x_n^{(n,a)}$ the zeros of $H_n^{\lambda_n}(x, a)$. Moreover

$$E_n^*(a) := \min_{(x_1, \dots, x_n) \in \Sigma_a^n} E(x_1, \dots, x_n) = E(x_1^{(n,a)}, \dots, x_n^{(n,a)}).$$

One of the interesting questions is to study the asymptotic of $E_n^*(a)$ as n goes to infinity. To do this, we consider on Σ_a the probability measure

$$\mu_n^a = \frac{1}{n} \sum_{i=1}^n \delta_{x_i^{(n,a)}}.$$

Using the general theory of logarithmic potential, we will prove that if $\lim_{n \rightarrow \infty} \frac{\lambda_n}{n} = \alpha \geq 0$, the measure μ_n^a scaled by the factor $\frac{1}{\sqrt{n}}$, converge tightly to some probability measure μ_α^a with support $S_{a,\alpha} \subset \Sigma_a$, which satisfies

$$\int_{S_{a,\alpha}} \log \frac{1}{|x - y|} \mu_\alpha^a(dy) + Q_\alpha(x) = \begin{cases} C & \text{if } x \in S_{a,\alpha} \\ \geq C & \text{if } x \notin S_{a,\alpha} \end{cases},$$

where $Q_\alpha(x) = x^2 + 2\alpha \log \frac{1}{|x|}$ and we give a explicit density to the measure above. Moreover we show that the asymptotic of the energy $E_n^*(a)$ is closely related to

$$E^*(a) := \int_{S_{a,\alpha}} \int_{S_{a,\alpha}} \log \frac{1}{|x-y|} \mu_\alpha^a(dy) \mu_\alpha^a(dx) + \int_{S_{a,\alpha}} Q_\alpha(x) \mu_\alpha^a(dx).$$

The measure μ_α^a is called the equilibrium measure, and $E^*(a)$ is the equilibrium energy. These results are closely related to those proved in [2] and [3].

In the first section we define the system of orthogonal polynomials in semi-finite interval of the form $[a, +\infty[$, and we give some simple asymptotic results for the coefficients of the the three terms recurrences relation. Moreover we give the second order differential equation satisfied by the truncated generalized Hermite polynomials which has been studied in [5], [10]. In section 2 we defined the electrostatic energy and we show that the minimum of the energy is attained at the zeros of the generalized Hermite polynomials. Furthermore we give the central result, which state the convergence of the global density of zeros to some probability density. This is the subject of theorem 2.3. We end the section by an asymptotic formula of the energy, where we prove that $E_n^*(a) = \mathbb{E}^* n^2 - n \left(\lambda_n + \frac{n}{2} \right) \log n + o(n^2)$, where \mathbb{E}^* is some constant and $o(x)$ is a small terms in x . For $\lambda_n = 0$, $E_n^*(a) \sim -\frac{n^2}{2} \log n$, where $a_n \sim b_n$ as $n \rightarrow +\infty$ means $a_n/b_n \rightarrow 1$ as $n \rightarrow +\infty$. Such result has been proved in [10], and references therein. Section 4 is the object to defined the truncated generalized Hermite polynomials where we restrict the weight to the symmetric set $]-\infty, a] \cup [a, +\infty[$ ($a \geq 0$), moreover we give some asymptotic of the coefficient of the three terms recurrences relation, which will be used later. As in the previous section we will give the global density of zeros when n goes to infinity, this is the object of theorem 5.2. In the last of section 3 and 4 we give some plots, which completes our analysis about the global density of zeros.

2 Half range generalized Hermite polynomials

Let $a \in \mathbb{R}$ and $\lambda > -\frac{1}{2}$. Consider on $[a, +\infty[$ the normalizing weight

$$w_\lambda(x) = \frac{1}{C_\lambda} |x|^{2\lambda} e^{-x^2},$$

where $C_\lambda = \int_a^{+\infty} |x|^{2\lambda} e^{-x^2} dx$. For $a = 0$, $C_\lambda = \frac{\Gamma(\lambda + \frac{1}{2})}{2}$.

On the space \mathcal{P} of polynomials in one variable with real coefficients we consider the inner product

$$\langle p, q \rangle = \int_a^{+\infty} p(x)q(x)w_\lambda(x)dx.$$

which makes \mathcal{P} into a preHilbert space. From the system $(t^m, m \in \mathbb{N})$ the Schmidt orthonormalizing produces a sequence $(H_m^{\lambda,a})_m$ of orthogonal polynomials: $H_m^{\lambda,a}$ is a polynomial of degree m and

$$\int_a^{+\infty} H_m^{\lambda,a}(x)H_n^{\lambda,a}(x)w_\lambda(x)dx = \delta_{nm}.$$

$H_m^{\lambda,a}$ is said the half range generalized Hermite polynomials.

We know that the sequence $H_n^{\lambda,a}$ satisfies three-terms recurrence relation $H_0^{\lambda,a}(x) = 1$, $H_1^{\lambda,a}(x) = \frac{(x - a_0)}{b_1}$,

$$xH_n^{\lambda,a}(x) = a_{n+1}H_{n+1}^{\lambda,a}(x) + b_nH_n^{\lambda,a}(x) + a_nH_{n-1}^{\lambda,a}(x),$$

where $b_n = \langle xH_n^{\lambda,a}, H_n^{\lambda,a} \rangle$ and $a_n = \langle xH_n^{\lambda,a}, H_{n-1}^{\lambda,a} \rangle$. Moreover if we write $H_n^{\lambda,a}(x) = \gamma_n x^n + \text{lower order terms}$, then $a_n = \frac{\gamma_{n-1}}{\gamma_n}$.

The sequences a_n, b_n and γ_n all depend in the parameters λ and a , but we omit this dependance in the notations.

Proposition 2.1 — for all $n \in \mathbb{N}$,

(1)

$$b_n \geq \frac{1}{2} (H_n^{\lambda,a}(a))^2 w_\lambda(a),$$

(2)

$$a_{n+1}^2 + a_n^2 + b_n^2 = n + \lambda + \frac{1}{2} + \frac{1}{2}a(H_n^{\lambda,a}(a))^2 w_\lambda(a),$$

moreover for n large enough and for all $\lambda \geq 0$, $b_n = O(\sqrt{n + \lambda})$ and, $(H_n^{\lambda,a}(a))^2 w_\lambda(a) = O(\sqrt{n + \lambda})$.

Proof.—

Step (1): By orthogonality we know

$$\int_a^{+\infty} H_n^{\lambda,a}(x)' H_n^{\lambda,a}(x) w_\lambda(x) dx = 0,$$

furthermore, integrate by part

$$\int_a^{+\infty} H_n^{\lambda,a}(x)' H_n^{\lambda,a}(x) w_\lambda(x) dx = -\frac{1}{2}(H_n^{\lambda,a}(a))^2 w_\lambda(a) + b_n - \frac{\lambda}{C_\lambda} \int_a^{+\infty} (H_n^{\lambda,a}(x))^2 x^{2\lambda-1} e^{-x^2} dx,$$

it follows that

$$b_n \geq \frac{1}{2}(H_n^{\lambda,a}(a))^2 w_\lambda(a).$$

Step (2): Integrate by part it follows that

$$\int_a^{+\infty} H_n^{\lambda,a}(x)' H_n^{\lambda,a}(x) x w_\lambda(x) dx = \langle x H_n^{\lambda,a}, x H_n^{\lambda,a} \rangle - (\lambda + \frac{1}{2}) \|H_n^{\lambda,a}\|^2 - \frac{1}{2} a (H_n^{\lambda,a}(a))^2 w_\lambda(a),$$

moreover

$$x H_n^{\lambda,a}(x)' = n H_n^{\lambda,a}(x) + c_{n-1} H_{n-1}^{\lambda,a}(x) + \dots + c_0 H_0^\lambda(x),$$

since

$$\int_a^{+\infty} H_n^{\lambda,a}(x)' H_n^{\lambda,a}(x) x w_\lambda(x) dx = \langle x (H_n^{\lambda,a})', H_n^{\lambda,a} \rangle = n \|H_n^{\lambda,a}\|^2 = n.$$

And from the recurrence relation

$$\langle x H_n^{\lambda,a}, x H_n^{\lambda,a} \rangle = a_{n+1}^2 + a_n^2 + b_n^2,$$

hence

$$a_{n+1}^2 + a_n^2 + b_n^2 = n + \lambda + \frac{1}{2} + \frac{1}{2} a (H_n^{\lambda,a}(a))^2 w_\lambda(a).$$

To see the boundedness of b_n , using step (1) and (2), it follows that

$$b_n^2 \leq n + \lambda + \frac{1}{2} + |a|b_n,$$

hence

$$(b_n - \frac{1}{2}|a|)^2 \leq n + \lambda + \frac{1}{2} + \frac{1}{4}a^2,$$

which means that

$$0 \leq b_n \leq \sqrt{n + \lambda + \frac{1}{2} + \frac{1}{4}a^2} + \frac{|a|}{2}.$$

And $\frac{b_n}{\sqrt{n + \lambda}}$ is bounded for all n . This complete the proof.

Proposition 2.2 — *The polynomials $H_n^{\lambda,a}$ satisfies the following differential equations*

(1)

$$H_n^{\lambda,a}(x)' = A_n(x)H_{n-1}^{\lambda,a}(x) - B_n(x)H_n^{\lambda,a}(x),$$

where

$$A_n(x) = 2a_n + \frac{2a_nb_n}{x} + \frac{a_nw_\lambda(a)(H_n^{\lambda,a}(a))^2}{x-a},$$

and

$$B_n(x) = \frac{a_nw_\lambda(a)H_n^{\lambda,a}(a)H_{n-1}^{\lambda,a}(a)}{x-a} + \frac{2a_n^2 - n - a_nw_\lambda(a)H_n^{\lambda,a}(a)H_{n-1}^{\lambda,a}(a)}{x}.$$

(2)

$$H_n^{\lambda,a}(x)'' + R_n(x)H_n^{\lambda,a}(x)' + S_n(x)H_n^{\lambda,a}(x) = 0,$$

where

$$R_n(x) = -2x + \frac{2\lambda}{x} - \frac{A_n'(x)}{A_n(x)},$$

and

$$S_n(x) = B_n'(x) - B_n(x)\frac{A_n'(x)}{A_n(x)} - B_n(x)\left(2x - \frac{2\lambda}{x} + B_n(x)\right) + \frac{a_n}{a_{n-1}}A_n(x)A_{n-1}(x).$$

See [1], [4] and [5] for the proof.

3 Density of zeros of half range Hermite polynomials

We propose that the weight function $w(x)$ creates two external fields. One is a long range field whose potential at a point $x > a$, is

$$-\log w(x) = x^2 + 2\lambda \log \frac{1}{x} + \log C_\lambda.$$

In addition in the presence of n unit charges, w produces a short range field whose potential is $\log\left(\frac{A_n(x)}{a_n}\right)$. Thus the total external potential $V(x)$ is the sum of the short and long range potentials, that is

$$V(x) = x^2 + 2\lambda \log \frac{1}{x} + \log C_\lambda + \log\left(2 + \frac{2b_n}{x} + \frac{w_\lambda(a)(H_n^{\lambda,a}(a))^2}{x-a}\right).$$

Consider the system of n movable unit charges in $[a; +\infty]^n$. In the presence of the external potential $V(x)$. If x_1, \dots, x_n are the position of the particles arranged in decreasing order. The total energy of the system is

$$F(x_1, \dots, x_n) = 2 \sum_{i < j} \log \frac{1}{|x_i - x_j|} + \sum_{i=1}^n V(x_i).$$

Proposition 3.1 — *The energy $F(x_1, \dots, x_n)$ is minimal at the zeros of half range Hermite polynomials $H_n^{\lambda,a}$.*

See for instance [10].

Let $x_1^{(n,a)}, \dots, x_n^{(n,a)} \in [a, +\infty[$ denote the zeros of half range generalized Hermite polynomials $H_n^{\lambda_n,a}$, where λ_n be some positive real sequence, and defined on $[a, +\infty[$ the probability measure

$$\mu_n^a = \frac{1}{n} \sum_{i=1}^n \delta_{\sigma_i^{(n,a)}},$$

where $\sigma_i^{(n,a)} = \frac{x_i^{(n,a)}}{\sqrt{n}}$ are the rescaled zeros. Moreover consider the modified rescaled energy in the sense

$$E(x_1, \dots, x_n) = F(x_1, \dots, x_n) - n \log C_{\lambda_n} = 2 \sum_{i < j} \log \frac{1}{|x_i - x_j|} + n \sum_{i=1}^n V_n(x_i),$$

where $V_n(x) = x^2 + 2\alpha_n \log \frac{1}{x} + \frac{1}{n} \log \left(2 + \frac{2b_n}{\sqrt{nx}} + \frac{\beta_n}{\sqrt{nx} - a} \right)$, $\alpha_n = \frac{\lambda_n}{n}$, and $\beta_n = w_{\lambda_n}(a)(H_n^{\lambda_n}(a))^2$. Then the minimum of the energy is attained at the rescaled zeros $\sigma_i^{(n,a)}$ of the polynomials $H_n^{\lambda_n, a}$.

Through all the rest of the paper it will be assumed that $a > 0$ if $\alpha \geq 0$, and $a \in \mathbb{R}$ if $\alpha = 0$, where $\alpha = \lim_{n \rightarrow +\infty} \alpha_n$.

Theorem 3.2 — Assume that the sequence α_n converges to some $\alpha \geq 0$. Then the measure μ_n^a converge for the tight topology to some probability measure μ_α^σ supported by $[\sigma, b]$ with density $f_{\alpha, \sigma}$, where $\sigma = \sigma(a, \alpha)$, and such that:

- (1) For $\alpha > 0$, and $a = 0$, there is a unique $\sigma_0 = \sigma(\alpha) > 0$, and a unique $b > \sigma_0$, and the density is

$$f_{\alpha, \sigma_0} = \frac{1}{\pi} \sqrt{(b-x)(x-\sigma_0)} \left(1 + \frac{\alpha}{\sqrt{\sigma_0 b}} \frac{1}{x} \right),$$

where

$$\sigma_0 = \sqrt{\frac{5}{3} + \frac{5\alpha}{3} - \frac{\beta}{3} - \frac{2}{3} \sqrt{2 + 4\alpha - 4\alpha^2 + 2(1+\alpha)\beta}},$$

$\beta = \sqrt{1 + 2\alpha + 4\alpha^2}$. And

$$b = b(\alpha) = \frac{2}{3} \left(\sqrt{6(\alpha+1) - 2\sigma_0^2} - \frac{\sigma_0}{2} \right).$$

- (2) For $\alpha \geq 0$, and $a > 0$, then $\sigma = a$, and

$$f_{\alpha, \sigma} = \frac{1}{2\pi} \sqrt{\frac{b-x}{x-\sigma}} \left(2x + b - \sigma - 2\alpha \sqrt{\frac{\sigma}{b}} \frac{1}{x} \right),$$

where $\sigma < b$, and b is the unique solution of the following equations

$$\frac{3}{4}(b-\sigma)^2 + \sigma(b-\sigma) + 2\alpha \sqrt{\frac{\sigma}{b}} - 2\alpha - 2 = 0, \text{ and } b + \sigma - \frac{2\alpha}{\sqrt{\sigma b}} \geq 0.$$

- (3) For $\alpha = 0$, and $a \in \mathbb{R}$, two cases are present:

- (a) if $a \geq -\sqrt{2}$, then $\sigma = a$.

(b) if $a < -\sqrt{2}$ then $\sigma = -\sqrt{2}$.

In each cases

$$f_{0,\sigma} = \frac{1}{2\pi} \sqrt{\frac{b-x}{x-\sigma}} (2x+b-\sigma),$$

and

$$b = \frac{2}{3} \left(\sqrt{\sigma^2 + 6} + \frac{\sigma}{2} \right).$$

From (3) if we put $\alpha = a = 0$, one obtains

$$f_{0,0}(x) = \frac{1}{2\pi} \sqrt{\frac{b-x}{x}} (2x+b),$$

where $b = \frac{2}{3}\sqrt{6}$. Such density appear in first time in [9], where the authors studies the probability that all eigenvalues of Gaussian Hermitian matrix are positives. A generalization of such question is treated in [2]. The author study the asymptotic density of eigenvalues in the interval $[a, +\infty[$ ($a \in \mathbb{R}$) for the Generalized Gaussian unitary ensemble matrices. He show that the probability that all eigenvalues of Generalized Gaussian Hermitian matrix to be positives is given by $f_{\alpha,a}$.

In the third case on can see that for $a \leq 0$, $f_{0,a}$ is a probability density, if $b \geq -a$, hence $\sqrt{a^2 + 6} \geq -2a$, which give that $a \geq -\sqrt{2}$. Which means that the limit case for which $f_{\alpha,a}$ is a probability density is $a \geq -\sqrt{2}$, and the limit density in that case is the semi-circle law

$$f_{0,-\sqrt{2}}(x) = \frac{1}{\pi} \sqrt{2-x^2}.$$

In this case $f_{0,-\sqrt{2}}$ is the distribution of zeros of the classical hermite polynomials H_n^0 . Moreover one can prove by potential theory that for all $a < -\sqrt{2}$, the density of zeros still unchanged, and is described by the semi-circle law. Which explain that the density of the zeros of H_n^0 , when we restricted the weight w_0 to the half line $[a, +\infty[$, for $a \leq -\sqrt{2}$, approximate the density of zeros of Hermite polynomials H_n , for the weight function w_0 on all the real line.

To prove the theorem we need some preliminary results.

Let $\mathfrak{M}^1(\Sigma)$ be the set of probability measures on the closed set $\Sigma \subset \mathbb{R}$. We equip the space $\mathfrak{M}^1(\Sigma)$ with the tight topology. For this topology a

sequence (ν_n) converge to a measure ν if, for every continuous bounded function f on Σ ,

$$\lim_{n \rightarrow \infty} \int_{\Sigma} f(x) \nu_n(dx) = \int_{\Sigma} f(x) \nu(dx).$$

This topology is metrizable. If Σ is bounded, then $\mathfrak{M}^1(\Sigma)$ is compact.

Let Σ be a closed set $\left(\Sigma = \mathbb{R},]-\infty, a], [a, +\infty[\text{ or }]-\infty, a] \cup [a, +\infty[\right)$, and Q a function defined on Σ with values on $]-\infty, +\infty]$, continuous on $\text{int}(\Sigma)$. If Σ is unbounded, it is assumed that

$$\lim_{|x| \rightarrow +\infty} \left(Q(x) - \log(1 + x^2) \right) = \infty.$$

If ν is a probability measure supported by Σ , the energy $E(\nu)$ of ν is defined by

$$\mathbb{E}(\nu) = \int_{\Sigma} U^{\nu}(x) \nu(dx) + \int_{\Sigma} Q(x) \nu(dx),$$

where

$$U^{\nu}(x) = \int_{\Sigma} \log \frac{1}{|x - t|} \nu(dt).$$

By a straightforward computation one can prove that $E(\nu)$ is bounded from below. Hence we defined

$$\mathbb{E}^* = \inf \left\{ \mathbb{E}(\nu) \mid \nu \in \mathfrak{M}^1(\Sigma) \right\}.$$

Theorem 3.3 — *If $\nu(dx) = f(x)dx$, where f is a continuous function with compact support $\subset \Sigma$. Then the potential $U^{\nu}(x) = \int_{\Sigma} \log \frac{1}{|x - t|} \nu(dt)$ is a continuous function, and, $\mathbb{E}^* \leq \mathbb{E}(\nu) < \infty$. Furthermore there is a unique measure $\nu^* \in \mathfrak{M}^1(\Sigma)$ such that*

$$\mathbb{E}^* = \mathbb{E}(\nu^*).$$

The support of ν^ is compact.*

This measure ν^ is called the equilibrium measure.*

Proposition 3.4 — *Let $\nu \in \mathfrak{M}^1(\Sigma)$ with compact support. Assume that the potential U^{ν} of ν is continuous and that there is a constant C such that*

- (i) $U^{\nu}(x) + \frac{1}{2}Q(x) \geq C$ on Σ .
- (ii) $U^{\nu}(x) + \frac{1}{2}Q(x) = C$ on $\text{supp}(\nu)$. Then ν is the equilibrium measure: $\nu = \nu^*$.

The constant C is called the (modified) Robin constant. Observe that

$$\mathbb{E}^* = C + \frac{1}{2} \int_{\Sigma} Q(x) \nu^*(dx).$$

See for the proofs of the previous theorem and proposition [7].

The proof of the next proposition can be found in [2].

Proposition 3.5 — *Consider the case where $\Sigma = [a, +\infty[$, $Q_{\alpha}(s) = s^2 + 2\alpha \log \frac{1}{s}$, with $\alpha \geq 0$. Then the equilibrium measure is μ_{α}^a of the theorem (2.2). Moreover for $\alpha = 0$, the energy is given by*

$$\mathbb{E}^* = \frac{1}{108} \left(81 + 72a^2 - 2a^4 + (30a + 2a^3) \sqrt{6 + a^2} - 108 \log \left(\frac{1}{6} (-a + \sqrt{6 + a^2}) \right) \right).$$

and for $\alpha = a = 0$,

$$\mathbb{E}^* = \frac{3}{4} + \frac{1}{2} \log 6.$$

For the proof of the value of the energy in the case $\alpha = 0$, see for instance [9].

Proof of theorem 2.2— We denote in the proof the equilibrium energy by

$$\mathbb{E}_{\alpha}^* = \mathbb{E}(\mu_{\alpha}^a).$$

Let defined

$$\tau_n = \frac{1}{n(n-1)} E_{\alpha_n, n}^* = \frac{1}{n(n-1)} E(\sigma_1^{(n,a)}, \dots, \sigma_n^{(n,a)}).$$

Consider for a probability measure μ the energy

$$\mathbb{E}_{\alpha_n}(\mu) = \int_a^{+\infty} \int_a^{+\infty} \log \frac{1}{|s-t|} \mu(ds) \mu(dt) + \int_a^{+\infty} Q_{\alpha_n}(s)(s) \mu(ds),$$

where $Q_{\alpha_n}(s) = s^2 + 2\alpha_n \log \frac{1}{s}$.

Since for a probability μ with support in $[a, +\infty[$

$$\begin{aligned} & \int_{[a, +\infty[^n} E(x_1, \dots, x_n) \mu(dx_1) \dots \mu(dx_n) \\ &= n(n-1) \int_a^{+\infty} \int_a^{+\infty} \log \frac{1}{|s-t|} \mu(ds) \mu(dt) + n^2 \int_a^{+\infty} Q_{\alpha_n}(s) \mu(ds) \\ &+ n \int_a^{+\infty} \log \left(2 + \frac{2b_n}{\sqrt{ns}} + \frac{\beta_n}{\sqrt{ns} - a} \right) \mu(ds). \end{aligned}$$

Hence

$$\begin{aligned}\tau_n \leq & \mathbb{E}_\alpha(\mu) + \frac{n^2}{n(n-1)} \int_a^{+\infty} (Q_{\alpha_n}(s) - Q_\alpha(s)) \mu(ds) + \frac{1}{n-1} \int_a^{+\infty} Q_\alpha(s) \mu(ds) \\ & + \frac{1}{n-1} \int_a^{+\infty} \log\left(2 + \frac{2b_n}{\sqrt{ns}} + \frac{\beta_n}{\sqrt{ns}-a}\right) \mu(ds).\end{aligned}$$

For $\mu = \mu_\alpha^a$ the equilibrium measure, one gets

$$\begin{aligned}\tau_n \leq & \mathbb{E}_\alpha^* + \frac{n^2}{n(n-1)} \int_a^{+\infty} (Q_{\alpha_n}(s) - Q_\alpha(s)) \mu_\alpha^a(ds) + \frac{1}{n-1} \int_a^{+\infty} Q_\alpha(s) \mu_\alpha^a(ds) \\ & + \frac{1}{n-1} \int_a^{+\infty} \log\left(2 + \frac{2b_n}{\sqrt{ns}} + \frac{\beta_n}{\sqrt{ns}-a}\right) \mu_\alpha^a(ds).\end{aligned}\tag{3.1}$$

Since

$$\left| \int_a^{+\infty} (Q_{\alpha_n}(s) - Q_\alpha(s)) \mu_\alpha^a(ds) \right| \leq 2C_0 |\alpha_n - \alpha|,$$

where $C_0 = \int_a^{+\infty} |\log(x)| \mu_\alpha^a(dx)$. Which implies that

$$\lim_{n \rightarrow +\infty} \int_a^{+\infty} (Q_{\alpha_n}(s) - Q_c(s)) \mu_\alpha^a(ds) = 0.\tag{3.2}$$

From proposition (2.1) and the fact that α_n converge then $\frac{b_n}{\sqrt{n}}$ and $\frac{\beta_n}{\sqrt{n}}$ are bounded by some positive constants c_2 and c'_2 , and then

$$0 \leq \frac{1}{n-1} \int_a^{+\infty} \log\left(2 + \frac{b_n}{\sqrt{ns}} + \frac{\beta_n}{\sqrt{ns}-a}\right) \mu_\alpha^a(ds) \leq \frac{1}{n-1} \int_a^{+\infty} \log\left(2 + \frac{2c_2}{s} + \frac{c'_2}{s-a}\right) \mu_\alpha^a(ds),$$

since the integral $\int_a^{+\infty} \log\left(2 + \frac{2c_2}{s} + \frac{c'_2}{s-a}\right) \mu_\alpha^a(ds)$ converge, hence

$$\lim_{n \rightarrow +\infty} \frac{1}{n-1} \int_a^{+\infty} \log\left(2 + \frac{b_n}{\sqrt{ns}} + \frac{\beta_n}{\sqrt{ns}-a}\right) \mu_{\alpha,a}(ds) = 0.\tag{3.3}$$

From equations (3.1), (3.2) and (3.3), one gets

$$\limsup_n \tau_n \leq \mathbb{E}_\alpha^*.\tag{3.4}$$

Let

$$k_n(s, t) = \log \frac{1}{|s-t|} + \frac{1}{2} V_n(s) + \frac{1}{2} V_n(t),$$

and

$$h_n(t) = V_n(t) - \log(1 + t^2),$$

where $V_n(x) = x^2 + 2\alpha_n \log \frac{1}{x} + \frac{1}{n} \log \left(2 + \frac{2b_n}{\sqrt{nx}} + \frac{\beta_n}{\sqrt{nx} - a} \right)$. The positive sequence α_n convergent, hence there is some positive constants $a_1, a_2 \geq 0$, such that $a_1 \leq \alpha_n \leq a_2$. From the fact that for all $x > a$, $\frac{1}{n} \log \left(2 + \frac{2b_n}{\sqrt{nx}} + \frac{\beta_n}{\sqrt{nx} - a} \right) \geq 0$ one gets,

$$h_n(t) \geq \begin{cases} t^2 + 2a_1 \log \frac{1}{t} - \log(1 + t^2) = h_1(t), & \text{if } a < t \leq 1 \\ t^2 + 2a_2 \log \frac{1}{t} - \log(1 + t^2) = h_2(t), & \text{if } t \geq 1. \end{cases}$$

Let

$$h(t) = \inf(h_1(t), h_2(t)),$$

and using the inequality

$$|s - t| \leq \sqrt{1 + s^2} \sqrt{1 + t^2},$$

it follows that, for all $n \in \mathbb{N}$ and all $s, t > 0$,

$$k_n(s, t) \geq \frac{1}{2}h(s) + \frac{1}{2}h(t).$$

and

$$\sum_{i \neq j} k_n(x_i, x_j) \geq (n-1) \sum_{i=1}^n h(x_i),$$

hence

$$\sum_{i \neq j} \log \frac{1}{|x_i - x_j|} + (n-1) \sum_{i=1}^n V_n(x_i) \geq (n-1) \sum_{i=1}^n h(x_i),$$

furthermore

$$V_n(x) \geq h(x),$$

hence

$$E_n^* = E\left(\sigma_1^{(n,a)}, \dots, \sigma_n^{(n,a)}\right) \geq n^2 \int_a^{+\infty} h(t) \mu_n^a(dt).$$

From (3.1) one gets,

$$\int_a^{+\infty} h(t) \mu_n^a(dt) \leq \frac{n(n-1)}{n^2} \tau_n \leq \frac{n-1}{n} (\mathbb{E}_\alpha^* + B_n),$$

where

$$B_n = \frac{n^2}{n(n-1)} \int_a^{+\infty} (Q_{\alpha_n}(s) - Q_\alpha(s)) \mu_\alpha^a(ds) + \frac{1}{n-1} \int_a^{+\infty} Q_\alpha(s) \mu_\alpha^a(ds) \\ + \frac{1}{n-1} \int_a^{+\infty} \log\left(2 + \frac{2b_n}{\sqrt{ns}} + \frac{\beta_n}{\sqrt{ns-a}}\right) \mu_\alpha^a(ds).$$

It has been seen in the previous that B_n goes to 0 as $n \rightarrow +\infty$, thus, there is some constant C_0 such that for all $n \in \mathbb{N}$,

$$\int_a^{+\infty} h(t) \mu_n^a(dt) \leq C_0.$$

Moreover $\lim_{t \rightarrow +\infty} h(t) = +\infty$, then by the Prokhorov criterium there is some subsequence $n_j \rightarrow \infty$ such that the measure $\mu_{n_j}^a$ converge for the tight topology to some probability measure σ^a . We will denote simply μ_n^a the subsequence.

For $\ell > 0$ consider the cut kernel $k_n^\ell(s, t) = \inf(k_n(s, t), \ell)$, and let

$$\widetilde{k}_{\alpha_n}(s, t) = \log \frac{1}{|s-t|} + \frac{1}{2} Q_{\alpha_n}(s) + \frac{1}{2} Q_{\alpha_n}(t),$$

where $Q_{\alpha_n}(s) = s^2 + 2\alpha_n \log \frac{1}{s}$, and $\widetilde{k}_n^\ell(s, t) = \inf(\widetilde{k}_n(s, t), \ell)$. It is easy to see that for all $n \in \mathbb{N}$, and all $s, t \in \mathbb{R}_+^*$,

$$\widetilde{k}_{\alpha_n}(s, t) \leq k_n(s, t),$$

and

$$\widetilde{k}_{\alpha_n}^\ell(s, t) \leq k_n^\ell(s, t). \quad (3.5)$$

Let $\varepsilon > 0$, there is n_0 , such that for all $n \geq n_0$,

$$\alpha - \varepsilon \leq \alpha_n \leq \alpha + \varepsilon,$$

Let $n \geq n_0$, divided $\mathbb{R}_+^2 \setminus \{(s, t) \mid s = t \text{ and } s = 0, \text{ and } t = 0\}$ to four region

$$R_1 = \{(s, t) \mid s \geq 1 \text{ and } t \geq 1\}, \quad R_2 = \{(s, t) \mid a < s \leq 1 \text{ and } a < t \leq 1\},$$

and

$$R_3 = \{(s, t) \mid a < s \leq 1 \text{ and } t \geq 1\}, \quad R_4 = \{(s, t) \mid s \geq 1 \text{ and } a < t \leq 1\}.$$

If $(s, t) \in R_1$, then

$$\widetilde{k_{\alpha_n}}(s, t) \geq \widetilde{k_{\alpha+\varepsilon}}(s, t).$$

If $(s, t) \in R_2$,

$$\widetilde{k_{\alpha_n}}(s, t) \geq \widetilde{k_{\alpha-\varepsilon}}(s, t).$$

If $(s, t) \in R_3$,

$$\widetilde{k_{\alpha_n}}(s, t) \geq \log \frac{1}{|s-t|} + \frac{1}{2}Q_{\alpha+\varepsilon}(t) + \frac{1}{2}Q_{\alpha-\varepsilon}(s),$$

hence

$$\widetilde{k_{\alpha_n}}(s, t) \geq \frac{1}{2}(\widetilde{k_{\alpha+\varepsilon}}(s, t) + \widetilde{k_{\alpha-\varepsilon}}(s, t)).$$

By symmetry of the kernel $\widetilde{k_{\alpha_n}}$ the last inequality is valid in R_4 .

We obtain in $\mathbb{R}_+^2 \setminus \{(s, t) \mid s = t \text{ and } s = 0, \text{ and } t = 0\}$,

$$\widetilde{k_{\alpha_n}}(s, t) \geq \theta_1 \widetilde{k_{\alpha+\varepsilon}}(s, t) + \theta_2 \widetilde{k_{\alpha-\varepsilon}}(s, t),$$

where $(\theta_1, \theta_2) = (1, 0)$ in R_1 , $(\theta_1, \theta_2) = (0, 1)$ in R_2 and $(\theta_1, \theta_2) = (\frac{1}{2}, \frac{1}{2})$ in $R_3 \cup R_4$.

Hence if we take the infimum then for all $(s, t) \in]a, +\infty[^2$,

$$\widetilde{k_{\alpha_n}}^\ell(s, t) \geq \theta_1 \widetilde{k_{\alpha+\varepsilon}}^\ell(s, t) + \theta_2 \widetilde{k_{\alpha-\varepsilon}}^\ell(s, t),$$

By making use of equation(3.5) it yields

$$k_n^\ell(s, t) \geq \theta_1 \widetilde{k_{\alpha+\varepsilon}}^\ell(s, t) + \theta_2 \widetilde{k_{\alpha-\varepsilon}}^\ell(s, t).$$

Take the energy in both sides of the previous inequality, it follows for all $n \geq n_0$ that

$$\theta_1 \mathbb{E}_{\alpha+\varepsilon}^\ell(\mu_n^a) + \theta_2 \mathbb{E}_{\alpha-\varepsilon}^\ell(\mu_n^a) \leq \mathbb{E}^\ell(\mu_n^a),$$

where \mathbb{E}^ℓ , is the truncated energy. Which gives

$$\theta_1 \mathbb{E}_{\alpha+\varepsilon}^\ell(\mu_n^a) + \theta_2 \mathbb{E}_{\alpha-\varepsilon}^\ell(\mu_n^a) \leq \frac{n(n-1)}{n^2} \tau_n - \frac{1}{n^2} \sum_{i=1}^n V_n(\sigma_i^{(n,a)}) + \frac{\ell}{n},$$

Since

$$V_n(x) \geq Q_{\alpha_n}(x),$$

and by a simple computation we can show that

$$\inf_{s \in \mathbb{R}_+} Q_{\alpha_n}(s) = \begin{cases} \alpha_n - \alpha_n \log \alpha_n, & \text{if } \alpha_n > 0 \text{ for some } n \\ 0 & \text{if } \alpha_n = 0 \forall n \end{cases}$$

Hence

$$\theta_1 \mathbb{E}_{\alpha+\varepsilon}^\ell(\mu_n^a) + \theta_2 \mathbb{E}_{\alpha-\varepsilon}^\ell(\mu_n^a) \leq \frac{n(n-1)}{n^2} \tau_n - \frac{1}{n} \alpha_n (1 - \log \alpha_n) + \frac{\ell}{n},$$

As n goes to infinity, using the fact that α_n converge we obtain

$$\liminf_n (\theta_1 \mathbb{E}_{\alpha+\varepsilon}^\ell(\mu_n^a) + \theta_2 \mathbb{E}_{\alpha-\varepsilon}^\ell(\mu_n^a)) \leq \liminf \tau_n,$$

hence

$$\theta_1 \mathbb{E}_{\alpha+\varepsilon}^\ell(\sigma^a) + \theta_2 \mathbb{E}_{\alpha-\varepsilon}^\ell(\sigma^a) \leq \liminf \tau_n,$$

applying the monotone convergence theorem, when ℓ goes to $+\infty$ we obtain

$$\theta_1 \mathbb{E}_{\alpha+\varepsilon}(\sigma^a) + \theta_2 \mathbb{E}_{\alpha-\varepsilon}(\sigma^a) \leq \liminf \tau_n,$$

Since $\theta_1 \mathbb{E}_{\alpha+\varepsilon}(\sigma^a) + \theta_2 \mathbb{E}_{\alpha-\varepsilon}(\sigma^a) = \mathbb{E}_\alpha(\sigma^a)$. It follows

$$\mathbb{E}_\alpha(\sigma^a) \leq \liminf \tau_n.$$

Furthermore

$$\mathbb{E}_\alpha^* = \inf_{\nu \in \mathfrak{M}^1(\mathbb{R})} \mathbb{E}_\alpha(\nu) \leq \mathbb{E}_\alpha(\sigma^a),$$

hence

$$\mathbb{E}_\alpha^* \leq \mathbb{E}_\alpha(\sigma^a) \leq \liminf \tau_n.$$

Therefore

$$\mathbb{E}_\alpha^* \leq \mathbb{E}_\alpha(\sigma^a) \leq \liminf \tau_n \leq \limsup \tau_n \leq \mathbb{E}_\alpha^*,$$

in the last inequality we have used equation (3.4). Then we obtain

$$\mathbb{E}_\alpha(\sigma^a) = \mathbb{E}_\alpha^* = \mathbb{E}_\alpha(\mu_\alpha^a).$$

This implies by unicity of the equilibrium measure that $\sigma^a = \mu_\alpha^a$. We have proved that μ_α^a is the only possible limit for a subsequence of the sequence

(μ_n^a) . It follows that the measure μ_n^a itself converges: for all bounded continuous function on $[a, +\infty[$,

$$\lim_{n \rightarrow \infty} \int_a^{+\infty} f(x) \mu_n^a(dx) = \int_a^{+\infty} f(x) \mu_\alpha^a(dx),$$

and

$$\lim_{n \rightarrow \infty} \tau_n = \mathbb{E}_\alpha^*.$$

■

Proposition 3.6 — For $a \geq 0$, let

$$E_n^*(a) = E\left(x_1^{(n,a)}, \dots, x_n^{(n,a)}\right),$$

be the minimum of the energy at the zeros of $H_n^{\lambda_n, a}$. If $\lim_{n \rightarrow +\infty} \frac{\lambda_n}{n} = 0$, then for n large enough

$$E_n^*(a) = \mathbb{E}^* n^2 - n\left(\lambda_n + \frac{n}{2}\right) \log n + o(n^2),$$

where \mathbb{E}^* is the energy given in proposition 3.5.

For $a = 0$, one obtains

$$E_n^* = \left(\frac{3}{4} + \frac{1}{2} \log 6\right) n^2 - n\left(\lambda_n + \frac{n}{2}\right) \log n + o(n^2).$$

For $\lambda_n = 0$, the last asymptotic formula gives an approximation of the energy in the case of half range Hermite polynomials.

$$E_n^* \sim -\frac{n^2}{2} \log n.$$

Such result is proved in [6].

More general for $\lim_{n \rightarrow +\infty} \frac{\lambda_n}{n} = \alpha > 0$, the energy \mathbb{E}_α^* , has been computed see for instance [2]. In this case one can easily obtains as $n \rightarrow +\infty$,

$$E_n^* = \mathbb{E}_\alpha^* n^2 - n\left(\lambda_n + \frac{n}{2}\right) \log n + o(n^2).$$

Moreover, it has been proved in [2], that for $a = 0$, and α small enough

$$\lim_{n \rightarrow +\infty} \left(\frac{1}{n^2} E_n^* + \left(\frac{\lambda_n}{n} + \frac{1}{2} \right) \log n \right) = \left(\frac{3}{4} + \frac{1}{2} \log 6 + C\alpha \right) + o(\alpha),$$

where $C = \frac{1}{432} \left(-36(-6 + \sqrt{6}) + (54 - 161\sqrt{6})\log 2 + 27(10 + \sqrt{6})\log 3 \right) \approx 0.6045$, and $o(\alpha)$ is a small term in α .

Proof.— We saw

$$E_n^*(a) = E(\sqrt{n}\sigma_1^{(n,a)}, \dots, \sqrt{n}\sigma_n^{(n,a)}),$$

hence

$$E_n^*(a) = \sum_{i < j} \log \frac{1}{|\sqrt{n}\sigma_i^{(n,a)} - \sqrt{n}\sigma_j^{(n,a)}|} + \sum_{i=1}^n V(\sqrt{n}\sigma_i^{(n,a)}) - n \log(C_{\lambda_n}),$$

since

$$V(\sqrt{nx}) = nx^2 + 2\lambda_n \log \frac{1}{\sqrt{nx}} + \log \left(2 + \frac{2b_n}{\sqrt{nx}} + \frac{\beta_n}{\sqrt{nx} - a} \right) + \log(C_{\lambda_n}),$$

hence

$$V(\sqrt{nx}) = nV_n(x) - \lambda_n \log n + \log(C_{\lambda_n}),$$

where $V_n(x) = \left(x^2 + 2\alpha_n \log \frac{1}{x} + \frac{1}{n} \log \left(2 + \frac{2b_n}{\sqrt{nx}} + \frac{\beta_n}{\sqrt{nx} - a} \right) \right)$. Hence

$$E_n^*(a) + \left(n\lambda_n + \frac{n(n-1)}{2} \right) \log n = E_{\alpha_n}^*,$$

By the previous theorem we get

$$\lim_{n \rightarrow +\infty} \frac{1}{n^2} \left(E_n^*(a) + \left(n\lambda_n + \frac{n(n-1)}{2} \right) \log n \right) = \lim_{n \rightarrow +\infty} E_{\alpha_n}^* = \mathbb{E}_\alpha,$$

Hence

$$\lim_{n \rightarrow +\infty} \left(\frac{1}{n^2} E_n^*(a) + \left(\frac{\lambda_n}{n} + \frac{1}{2} \right) \log n \right) = \mathbb{E}_\alpha,$$

and the case $\alpha = 0$, gives the desired result. ■

Plotting of density of zeros.

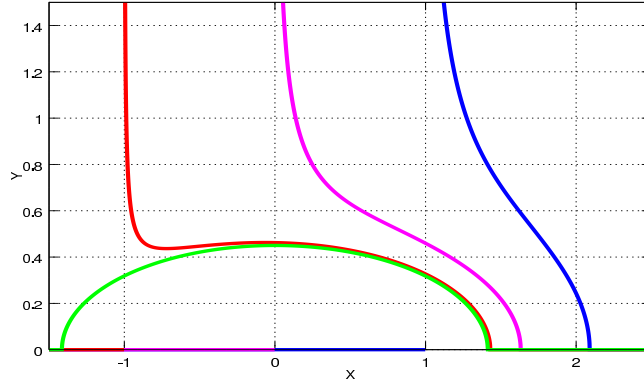


Figure 1: Plot of $f_{\alpha,a}$ for $\alpha = 0$

- for $a = 1$, $b \approx 2.09$, $\sigma = 1$, density of eigenvalues in $\Sigma_\sigma = [1, +\infty[$.
- for $a = 0$, $b = \frac{2}{3}\sqrt{6} \approx 1.632$, $\sigma = 0$, density of zeros in $\Sigma_\sigma = [0, +\infty[$.
- for $a = -1$, $b = 1.43$, $\sigma = -1$, density of zeros in $\Sigma_\sigma = [-1, +\infty[$.
- for $a = -\sqrt{2}$, $b = \sqrt{2}$, $\sigma \leq -\sqrt{2}$, density of zeros in $\Sigma_\sigma = [\sigma, +\infty[$.

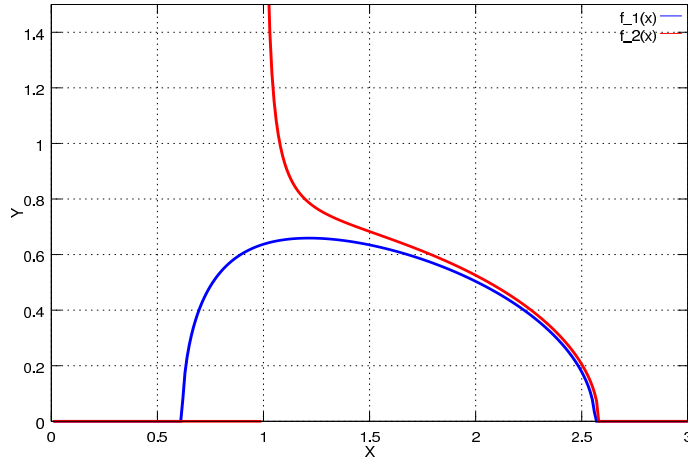


Figure 2: Plot of $f_{\alpha,a}$ for $\alpha = 2$.

- $a = 1$, $b = 2.58$, density of zeros in $[1, +\infty[$.
- $a = a_c \approx 0.618$, $b = b_c \approx 2.562$, density of zeros in $]a, +\infty[$, for $0 \leq a \leq a_c$

4 Truncated Generalized Hermite polynomials

Let $a \in \mathbb{R}$, and $\lambda \geq 0$. Consider in $] -\infty, -a] \cup [a, +\infty[$, the probability measure

$$w_\lambda(x) = \frac{1}{C_{\lambda,a}} |x|^{2\lambda} e^{-x^2} = e^{-v_\lambda(x)},$$

where $C_{\lambda,a} = 2 \int_a^{+\infty} |x|^{2\lambda} e^{-x^2} dx$. For $a = 0$, $C_\lambda = \Gamma(\lambda + \frac{1}{2})$. As in the first section to the weight function w_λ we associate the sequence of orthonormalizing polynomials $H_n^{\lambda,a}$. They satisfies

$$\int_{]-\infty, -a] \cup [a, +\infty[} H_n^{\lambda,a}(x) H_m^{\lambda,a}(x) w_\lambda(x) dx = \delta_{nm}.$$

By the symmetry of the weight they satisfies the three-terms recurrence relation $H_0^{\lambda,a}(x) = 1$, $H_1^{\lambda,a}(x) = \frac{x - a_1}{b_1}$ and for all $n \geq 1$,

$$x H_n^{\lambda,a}(x) = a_{n+1} H_{n+1}^{\lambda,a}(x) + a_n H_{n-1}^{\lambda,a}(x).$$

Moreover if we write $H_n^{\lambda,a}(x) = \gamma_n x^n + \text{lower order terms}$, then $a_n = \frac{\gamma_{n-1}}{\gamma_n}$.

The sequences a_n, b_n and γ_n all depend in the parameters λ and a , but we omit this dependance in the notations.

Proposition 4.1 — *Let $b_n = \max(|a_n|, |a_{n+1}|)$, then*

$$a_n^2 + a_{n+1}^2 = n + \lambda + \frac{1}{2} + a(H_n^{\lambda,a}(a))^2 w_\lambda(a),$$

and for $a \neq 0$,

$$(H_n^{\lambda,a}(a))^2 w_\lambda(a) = O(b_n^2).$$

Moreover

$$\lim_{n \rightarrow +\infty} b_n^2 = +\infty.$$

If an addition $a = 0$ then $a_n = O(\sqrt{n + \lambda})$.

Proof.—By orthogonality we have

$$n = \int_{]-\infty, -a] \cup [a, +\infty[} x H_n^{\lambda,a}(x)' H_n^{\lambda,a}(x) w_\lambda(x) dx,$$

Integrate by part we obtain

$$n = -a(H_n^{\lambda,a}(a))^2 w_\lambda(a) - \frac{1}{2} - \frac{1}{2} \int_{]-\infty, -a] \cup [a, +\infty[} x(H_n^{\lambda,a}(x))^2 w'_\lambda(x) dx,$$

Since

$$w'_\lambda(x) = \left(-2x + \frac{2\lambda}{x}\right) w_\lambda(x),$$

it follows that

$$\int_{]-\infty, -a] \cup [a, +\infty[} x(H_n^{\lambda,a}(x))^2 w'_\lambda(x) dx = -2(xH_n^{\lambda,a} | xH_n^{\lambda,a}) + 2\lambda(H_n^{\lambda,n} | H_n^{\lambda,a}),$$

From the recurrence relation one gets,

$$n = -a(H_n^{\lambda,a}(a))^2 w_\lambda(a) - \frac{1}{2} + a_n^2 + a_{n+1}^2 - \lambda,$$

and the result follow. For the second step one can see that

$$\frac{(H_n^{\lambda,a}(a))^2 w_\lambda(a)}{b_n^2} \leq \frac{1}{a},$$

moreover, $b_n^2 \geq n+1$, hence $\lim_{n \rightarrow +\infty} b_n = +\infty$.

If $a = 0$, then

$$a_n^2 + a_{n+1}^2 = 2n + 1 + 2\lambda,$$

and

$$|a_n| \leq \sqrt{2n + 2\lambda + 1}.$$

Which gives the desired result.

Proposition 4.2 — *The polynomials satisfies the differential equations*

(1)

$$H_n^{\lambda,a}(x)' = A_n(x)H_{n-1}^{\lambda,a}(x) - B_n(x)H_n^{\lambda,a}(x),$$

where

$$A_n(x) = 2a_n \left(1 + \frac{aH_n^{\lambda,a}(a)^2 w_\lambda(a)}{x^2 - a^2}\right),$$

and

$$B_n(x) = 2a_n H_n^{\lambda,a}(a) H_{n-1}^{\lambda,a}(a) \frac{x}{a^2 - x^2} + \frac{2\lambda a_n}{x} \int_{|y| \geq a} H_n^{\lambda,a}(y) H_{n-1}^{\lambda,a}(y) w_\lambda(y) \frac{dy}{y}.$$

(2)

$$H_n^{\lambda,a}(x)'' + R_n(x)H_n^{\lambda,a}(x)' + S_n(x)H_n^{\lambda,a}(x) = 0,$$

where

$$R_n(x) = -2x + \frac{2\lambda}{x} - \frac{A_n'(x)}{A_n(x)},$$

and

$$S_n(x) = B_n'(x) - B_n(x) \frac{A_n'(x)}{A_n(x)} - B_n(x) \left(2x - \frac{2\lambda}{x} + B_n(x) \right) + \frac{a_n}{a_{n-1}} A_n(x) A_{n-1}(x)$$

5 Density of zeros of truncated symmetric generalized Hermite polynomials

Let

$$V(x) = x^2 + 2\lambda \log \frac{1}{|x|} + \log C_\lambda + \log \left(2 + 2w_\lambda(a) (H_n^{\lambda,a}(a))^2 \frac{a}{x^2 - a^2} \right)$$

Consider the system of n movable unit charges in $\left(] - \infty, -a[\cup] a, +\infty[\right)^n$ in the presence of the external potential $V(x)$. If x_1, \dots, x_n are the position of the particles arranged in decreasing order. The total energy of the system is

$$F(x_1, \dots, x_n) = 2 \sum_{i < j} \log \frac{1}{|x_i - x_j|} + \sum_{i=1}^n V(x_i),$$

Proposition 5.1 — *The energy $F(x_1, \dots, x_n)$ is minimal at the zeros of the truncated generalized Hermite polynomials $H_n^{\lambda,a}$.*

See for instance [10].

Let $x_1^{(n,a)}, \dots, x_n^{(n,a)} \in] - \infty, -a[\cup] a, +\infty[$ denote the zeros of truncated generalized Hermite polynomials $H_n^{\lambda_n,a}$, where λ_n be some positive real sequence, and defined on $] - \infty, -a[\cup] a, +\infty[$ the probability measure

$$\nu_n^a = \frac{1}{n} \sum_{i=1}^n \delta_{\sigma_i^{(n,a)}},$$

where $\sigma_i^{(n,a)} = \frac{x_i^{(n,a)}}{b_n}$ are the rescaled zeros. Moreover consider the modified rescaled energy in the sense

$$E(x_1, \dots, x_n) = F(x_1, \dots, x_n) - n \log C_{\lambda_n} = 2 \sum_{i < j} \log \frac{1}{|x_i - x_j|} + b_n \sum_{i=1}^n V_n(x_i),$$

where $V_n(x) = x^2 + 2\alpha_n \log \frac{1}{x} + \frac{1}{b_n} \log \left(2 + \frac{2a\beta_n}{b_n^2 x^2 - a^2} \right)$, $\alpha_n = \frac{\lambda_n}{b_n}$, and $\beta_n = w_{\lambda_n}(a)(H_n^{\lambda_n}(a))^2$. Then the minimum of the energy is attained at the rescaled zeros $\sigma_i^{(n,a)}$ of the polynomials $H_n^{\lambda_n, a}$. For $a = 0$ one can take instead of b_n the sequence n .

Theorem 5.2 — *We assume that, the sequence α_n converge to some $\alpha \geq 0$. Then for all $a \geq 0$, the measure ν_n^a converge for the tight topology to some probability measure ν_α^σ supported by $[-b, -\sigma] \cup [\sigma, b]$ with density $f_{\alpha, \sigma}$, such that*

- (1) *If $a = 0$ and $\alpha > 0$, then there is a unique $\sigma_0 = \sigma(\alpha) > 0$ and a unique $b_0 = b(\alpha)$ such that $b > \sigma_0 > 0$, and*

$$f_{\alpha, \sigma_0}(x) = \frac{1}{\pi|x|} \sqrt{(b_0^2 - x^2)(x^2 - \sigma_0^2)},$$

where

$$b_0 = \sqrt{1 + \alpha + \sqrt{1 + 2\alpha}},$$

and

$$\sigma_0 = \sqrt{1 + \alpha - \sqrt{1 + 2\alpha}}.$$

- (2) *If $0 < a \leq \sigma_0$ and $\alpha > 0$, then $\sigma(a, \alpha) = \sigma_0$ and the density on Σ still the same as in (1).*

- (3) *If $a > \sigma_0$ and $\alpha > 0$, then $\sigma(a, \alpha) = a$ and,*

$$f_{\alpha, \sigma}(x) = \frac{1}{\pi|x|} \sqrt{\frac{b^2 - x^2}{x^2 - \sigma^2}} \left(x^2 - \frac{\alpha\sigma}{b} \right).$$

where $b = b(\alpha, \sigma)$ is the unique solutions of the following equations

$$\sigma b - \alpha \geq 0, \text{ and } \frac{b^2}{2} + \frac{\alpha\sigma}{b} - \frac{\sigma^2}{2} - \alpha - 1 = 0.$$

(4) If $a \geq 0$ and $\alpha = 0$, then $\sigma(a, 0) = a$ and,

$$f_{0,\sigma}(x) = \frac{|x|}{\pi} \sqrt{\frac{2 + \sigma^2 - x^2}{x^2 - \sigma^2}},$$

$$\text{where } b = \sqrt{\sigma^2 + 2}$$

One recover's from (1) or (4) the density of zeros of classical Hermite polynomials which is given by the semi-circle law

$$f_{0,0}(x) = \frac{1}{\pi} \sqrt{2 - x^2}.$$

Moreover using mathematica one can find the explicit value of b in (4), but it is very complicate to put there here.

The proof of theorem 4.2 follow as theorem 2.2 with slight modifications.

As in the previous section, in [2] the energy $E_{\alpha,a}^*$ of the measure ν_α^a in the previous theorem has been computed, and as in the previous section one can prove that, the asymptotic of the equilibrium energy of the zeros is as follows:

$$E_n^*(a) = E_{\alpha,a}^* n^2 - n \left(\lambda_n + \frac{n}{2} \right) \log n + o(n^2).$$

For the particular case, if $\lim_{n \rightarrow +\infty} \frac{\lambda_n}{n} = \alpha = 0$, one gets

$$E_n^*(a) = \left(\frac{3}{4} + \frac{1}{2} \log 2 + a^2 \right) n^2 - n \left(\lambda_n + \frac{n}{2} \right) \log n + o(n^2),$$

Plotting of the density of zeros $f_{\alpha,\sigma}$

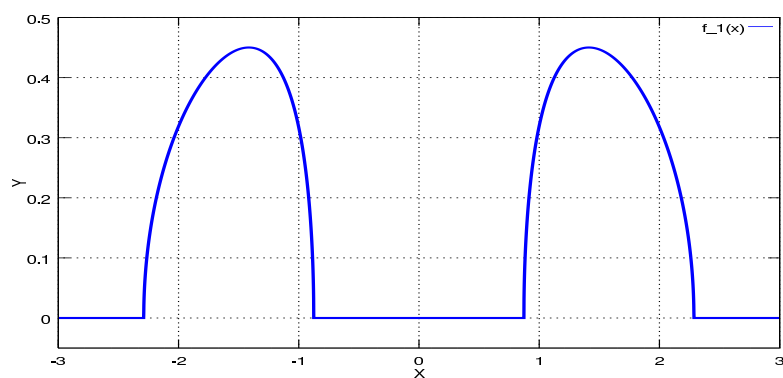


Figure 3: $\alpha = 2$, $\sigma = a_c \approx 0.874$, $b = b_c \approx 2.2882$
Density of zeros on $] -\infty, -a] \cup [a, +\infty[$, for all $0 < a \leq 0.874$.

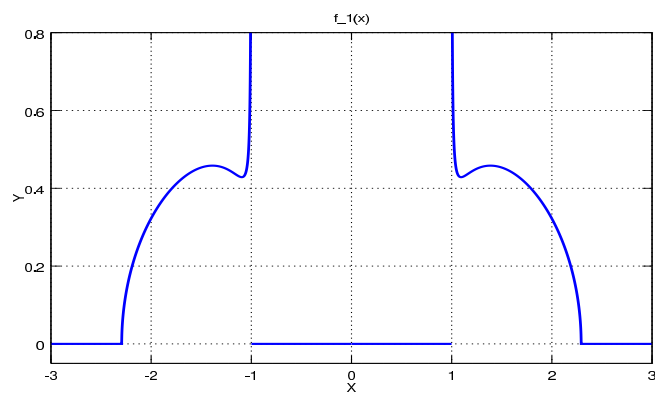


Figure 4: $\alpha = 2$, $a = 1 > a_c$, $b = 2.2924$
Density of zeros on $] -\infty, -1] \cup [1, +\infty[$.

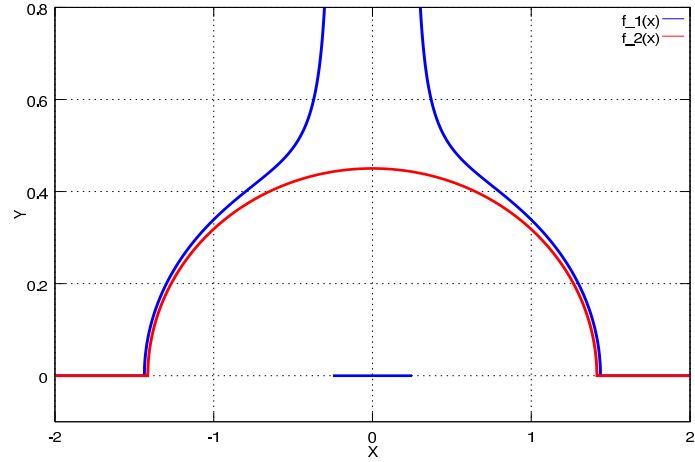


Figure 5: $\alpha = 0$

- $a = a_c = 0, b = b_c = \sqrt{2}$ semi-circle law, density of zeros on \mathbb{R} .
— $a = 0.25, b = 1.436$ density of zeros on $] -\infty, -0.25] \cup [0.25, +\infty[$.

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